Long Integral

$$* = \int_0^\infty \arctan^2\left(\frac{1}{x}\right) dx$$

let $\arctan\left(\frac{1}{x}\right) = u$

$\therefore \frac{1}{\left(\frac{1}{x}\right)^2 + 1} \left(\frac{-1}{x^2}\right) dx = du$

$\therefore \frac{x^2}{x^2+1} \cdot \frac{-1}{x^2} \, dx = \partial u$

$\therefore \frac{-1}{x^2+1} \, dx = du$

$dx = -(x^2+1) \, du$

and $\frac{1}{x} = \tan(u)$

$\therefore x = \cot(u)$

$\therefore x^2+1 = \cot^2(u)+1$

hence $*$ is

$$\int_{\pi/2}^{0} - u^2(\cot^2(u)+1) \, du$$

$$\Rightarrow \int_0^{\pi/2} u^2(\cot^2(u)+1) \, du$$

note

$\cot^2(u)+1 = \csc^2(u)$

$$\therefore \Rightarrow \int_0^{\pi/2} u^2(\csc^2(u)) \, du$$

IBP:

let ~~dv~~ $v = u^2$ $\qquad \frac{du}{dx} = \csc^2(u)$

$\frac{dv}{dx} = 2u$ $\qquad u = -\cot(u)$

$$\therefore \underbrace{-u^2\cot(u)\Big|_0^{\pi/2}} + \underbrace{2\int_0^{\pi/2} u\cot(u)\,du}_{(\dagger)}$$

$$-\left(\frac{\pi}{2}\right)^2\cot\left(\frac{\pi}{2}\right) + 0$$

$$\Rightarrow \boxed{0}$$

$$(\dagger) \Rightarrow 2\int_0^{\pi/2} u\cot(u)\,du$$

let $v = u$ $\qquad \frac{du}{dx} = \overset{\cot(u)}{\cancel{\ln|\sin(x)|}}$

$\frac{dv}{dx} = 1$ $\qquad u = \ln|\sin(x)|$

$$\therefore (\dagger) = 2\left[\underbrace{u\ln|\sin(x)|\Big|_0^{\pi/2}}_{0} - \underbrace{\int_0^{\pi/2}\ln(\sin(x))|\,dx}_{(\ddagger)}\right]$$

$$(\ddagger) = \int_0^{\pi/2}\ln|\sin(x)|\,dx$$

let $u = \frac{\pi}{2} - x$ $\qquad$ dummy variable.

$$\therefore -\int_{\cancel{0}-\pi/2}^{0}\ln|\cos(\overset{u}{\cancel{x}})|\,d\overset{u}{\cancel{x}} = (\ddagger) = \int_0^{\pi/2}\ln|\sin(x)|\,dx$$

$$\therefore I = \int_0^{\pi/2}\ln|\cos(x)|\,dx = \int_0^{\pi/2}\ln(\sin(x))|\,dx$$

Therefore $2I = \int_0^{\pi/2} \ln(\sin(x)) + \ln(\cos(x))\, dx$

$$= \int_0^{\pi/2} \ln(\sin(x)\cos(x))\, dx$$

$$= \int_0^{\pi/2} \ln\left(\tfrac{1}{2}\sin(2x)\right) dx$$

$$2I = \int_0^{\pi/2} \ln(\sin(2x)) - \ln(2)\, dx$$

Let $m = 2x$

$\therefore$

$$2I = \frac{1}{2}\int_0^{\pi} \ln\sin(m) - \ln(2)\, dm$$

Let $v = \frac{\pi}{2} - m$

$$\therefore 2I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \ln|\cos(v)|dv - \frac{1}{2}\int_0^{\pi} \ln(2)\, dm$$

$$= \frac{1}{2}(2)\int_0^{\pi/2} \ln|\cos(v)|dv - \frac{1}{2}\int_0^{\pi} \ln(2)\, dm.$$

$$= \int_0^{\pi/2} \ln|\cos(v)|dv - \frac{1}{2}\int_0^{\pi} \ln(2)\, dm$$

$$2I = I - \frac{1}{2}\int_0^{\pi} \ln(2)\, dm$$

$$\therefore I = -\frac{1}{2}\int_0^{\pi} \ln(2)\, dm$$

$$= \frac{-\pi}{2}\ln(2).$$

$$\therefore (+) = 2\left(0 - \left(-\frac{\pi}{2}\ln(2)\right)\right)$$

$$= 2\left(\frac{\pi}{2}\ln(2)\right)$$

$$= \underline{\pi \ln(2)}.$$