BMO2 question

$\sqrt{a} + \sqrt{b} = \sqrt{2009}$ $(*)$ (find all non negative integers a, b s.t equation is satisfied)

NOTE

- a, b are interchangeable so a distinct solution (a, b) and yield (b, a) due to the ~~sa~~ symmetry of the equation

- $2009 = 49 \cdot 41$

- Squaring both sides of $(*)$ yields
$a + b + 2\sqrt{ab} = 2009$

$\because$ RHS = integer $\therefore$ LHS = integers but $a, b \in \mathbb{Z}^+$
$\therefore$ $2\sqrt{ab} \in \mathbb{Z}$ hence $\sqrt{ab} \in \mathbb{Z}^+$
thus $ab = n^2$ where $n \in \mathbb{Z}^+$.

$\therefore$ $a + b = 2009 - 2n$

hence $a^2 + 2ab + b^2 = 2009^2 - 4(2009n) + 4n^2$

$\therefore$ $a^2 + 2n^2 + b^2 = 2009^2 - 4(2009n) + 4n^2$

$\therefore$ $a^2 - 2n^2 + b^2 = 2009(2009 - 4n)$

$\therefore$ $(a-b)^2 = 2009(2009 - 4n)$

- Recall that $2009 = 41 \cdot 49$
hence $(a-b)^2 = 7^2 \cdot 41 (\cancel{20}\, 7^2 \cdot 41 - 4n)$

Analysing the RHS, the value in bracket must be of the form $41 \cdot v^2$ ($v \in \mathbb{N}$) such that $(a-b) = (41 \cdot 7 \cdot v)^2$ (a square power of 7 is available but only 1 power of 41 is available)

Hence, $2009 - 4n = 41 \cdot v^2$

$\therefore \quad 49 - \frac{4}{41} n = v^2$

$\because$ RHS $\in \mathbb{N} \therefore$ LHS $\in \mathbb{N} \therefore \frac{4}{41} n \in \mathbb{N}$

hence n is a multiple of 41.

Thus, by setting n to be a multiple of 41, we obtain a number that has a multiple of 4 difference and is a square number.

Hence, the only solutions are

$49 - 4(6) = 25$ (case 1)

$49 - 4(10) = 9$ (case 2)

$49 - 4(12) = 1$ (case 3)

Case 1:

$6 = \frac{n}{41} \therefore n = 6 \cdot 41 \quad = \sqrt{ab} \therefore ab = 36 \cdot 41^2$

$\therefore a = \frac{36 \cdot 41^2}{b}$

$v = 5$

$\therefore (a-b)^2 = (41 \cdot 7 \cdot 5)^2$

$\therefore a - b = 41 \cdot 7 \cdot 5$

$\therefore \frac{36 \cdot 41^2}{b} - b = 41 \cdot 7 \cdot 5$

$$\therefore \frac{36 \times 41^2}{b} - b = 35 \times 41$$

$$\therefore 36 \times 41^2 - b^2 = 35 \times 41\, b$$

$$\therefore b^2 + 35 \times 41\, b - 36 \times 41^2 = 0$$

$$\therefore (b + 36 \times 41)(b - 41)$$

hence $b = 41$ Bcos $b > 0$

$$\therefore a = 36 \cdot 41$$

Case 2:

$$\frac{10}{} = \frac{n}{41} \quad \therefore 41 \cdot 10 = n = \sqrt{ab}$$

$$\therefore 100 \cdot 41^2 = ab$$

hence $a = \frac{100 \cdot 41^2}{b}$

$$(a-b)^2 = (7 \cdot 3 \cdot 41)^2$$

$$\therefore a - b = 21 \cdot 41$$

$$\frac{100 \cdot 41^2}{b} - b = 21 \cdot 41$$

$$\therefore b^2 + 21 \cdot 41\, b - 100 \cdot 41^2 = 0$$

$$\therefore (b + 25 \cdot 41)(b - 4 \cdot 41) = 0$$

$$\therefore b = 4 \cdot 41$$

$$a = \underline{25 \cdot 41}$$

Case 3:

$\cancel{(a-b)^2 = (7 \cdot 41)}$ $12 = \frac{n}{41}$ $\therefore (12 \cdot 41)^2 = ab$

$$\Rightarrow 144 \cdot 41^2$$

$$\therefore a = \frac{144 \cdot 41^2}{b}$$

$$\therefore (a-b)^2 = (7 \cdot 41)^2$$

$$\therefore a - b = 7 \cdot 41$$

$$\frac{144\cdot 41^2}{b} - b = 7\cdot 41$$

$$b^2 + 7\cdot 41 b - 144\cdot 41^2 = 0$$

$$\therefore \ (b + 16\cdot 41)(b - 9\cdot 41) = 0$$

$$b = 9\cdot 41$$

$$a = 16\cdot 41$$

$\therefore$ Solotion one : $(16\times 41, 9\times 41)$

$(9\times 41, 16\times 41)$

$(4\times 41, 25\times 41)$

$(25\times 41, 4\times 41)$

$(36\times 41, 41)$

$(41, 36\times 41)$