nice integral

$$\int_0^{\pi} (1+2x)\frac{\sin^3(x)}{1+\cos^2(x)}\,dx$$

let $f(\sin(x))$ be some function taking in $\sin(x)$ as input

consider $\int_0^{\pi} x f(\sin(x))\,dx = I$

let $u = \pi - x$

$$\therefore \int_{\pi}^{0} -(\pi-u) f(\sin(\pi-u))\,du$$

$$\Rightarrow \int_0^{\pi} (\pi-u) f(\sin(\pi-u))\,du$$

$$\Rightarrow \int_0^{\pi} (\pi-u) f((\sin u))\,du = I$$

as $\sin(\pi-u) \equiv \sin(u)$

hence $\because$ u is a dummy variable.

$$2I = \int_0^{\pi} (x + (\pi - x)) f(\sin(x))\,dx$$

$$2I = \int_0^{\pi} \pi f(\sin(x))\,dx$$

$$\therefore I = \frac{\pi}{2}\int_0^{\pi} f(\sin(x))\,dx$$

$$f(\sin(x)) = \frac{\sin^3(x)}{1+\cos^2(x)}.$$

Original integral

$$\Rightarrow \int_0^\pi \frac{\sin^3(x)}{1+\cos^2(x)}\,dx + 2\int_0^\pi x\left(\frac{\sin^3(x)}{1+\cos^2(x)}\right)dx$$

$$\Rightarrow \int_0^\pi \frac{\sin^3(x)\,dx}{1+\cos^2(x)} + \pi\int_0^\pi \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$$

$$\Rightarrow (1+\pi)\int_0^\pi \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$$

Let $\cos(x) = u$

$$-\sin(x)\,dx = du$$

$$dx = -\frac{1}{\sin(x)}\,du$$

$$\therefore \Rightarrow (1+\pi)\int_1^{-1} -\frac{\sin^2(x)}{1+u^2}\,du$$

$$\Rightarrow (1+\pi)\left(\int_{-1}^{1} \frac{1-u^2}{1+u^2}\,du\right)$$

$$\Rightarrow (1+\pi)\left(\int_{-1}^{1} -1 + \frac{2}{1+u^2}\,du\right)$$

$$\Rightarrow (1+\pi)\left(2\int_{-1}^{1} \frac{1}{1+u^2}\,du - 2\right)$$

$$\Rightarrow (1+\pi)(2)\left(\arctan(u)\Big|_{-1}^{1} - 1\right)$$

$$\Rightarrow (1+\pi)(2)\left(\frac{\pi}{2}-1\right)$$

$$\Rightarrow (\pi+1)(\pi-2)$$