Q7.

~~log~~) let $x = 1/z$ where z is positive.

$\therefore$ $\cancel{x}$

$\therefore \quad \left(\frac{1}{z}\right)^{1/z} = x^x$

$\therefore \quad \log_z\left(\left(\frac{1}{z}\right)^{1/z}\right) = 1/z$

and $\log_z(1/z) = -1$

and since $1/z > -1$

thus $\log_z\left(\left(\frac{1}{z}\right)^{1/z}\right) > \log_z(1/z)$

hence $x^x > x$ for $0 < x < 1$

for $x > 1$ $\quad x > x^x$

$\therefore \quad 0 < x^x < 1$

$\therefore \quad x < f(x) < 1$ ~~for~~ as given by previous work

$\therefore \quad -1 < -f(x) < -x$

$\therefore \quad \cancel{(1/x)^{-1} < \left((1/x)^{-1}\right)^{-f(x)} < (1/x)^{-x}}$ (for $0 < x < 1$)

$\therefore \quad \cancel{< f(x)}$

$\therefore \log_{1/x}(x) < f(x)\log_{1/x}(x) < \log_{1/x}(f(x))$

for $0 < x < 1$ as $\underline{1/x > 1}$ $\quad \therefore x < x^{f(x)} < f(x)$

$\therefore x < g(x) < f(x)$ for $0 < x < 1$

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and $\log_x(x^x) < f(x)\log_x(x) < \log_x(x)$

$\therefore \quad f(x) < g(x) < x$ for $x > 1$

This is for $x > 1$ as the logarithm operators are now justified.

(iii) $x = 1/z$

$\therefore \quad z^{-1/z} = e^{-1/z \ln(z)}$

$\therefore \quad f'(x) = \left(-\frac{\ln(z)}{z}\right)' x^x$

$= -\left(\underset{\downarrow}{1/z}\ \underset{\downarrow}{\ln(z)}\right) x^x$

$\left(-1/z^2 \quad 1/z\right)$

$= -\left(1/z^2 - \frac{\ln(z)}{z^2}\right) x^x$

$= \frac{\ln(z)-1}{z^2} x^x$

hence $\ln(z) - 1 = 0$

$\therefore \ln(z) = 1$

$z = e$

$\therefore x = 1/e$.

(ii) $\lim_{x \to 0} f(x) = y$

$\therefore \quad \ln(y) = \lim_{x \to 0} \ln(f(x))$

$= \lim_{x \to 0} x \ln(x)$

$= 0$

$\therefore y \Rightarrow e^0 \Rightarrow 1$

and $\lim_{x \to 0} g(x) \Rightarrow 1$ as well.

(iv) for $x \geq 1$

$\ln(x) \geq 0$

$\therefore \int \ln(x) \geq \int 0$

$\therefore x\ln(x) - x \geq C \qquad (a+C = -1)$

$\therefore x\ln(x) - x \geq -1$

$\therefore x\ln(x) + 1 \geq x$

$\therefore \ln(x) + 1/x \geq 1$

for $0 < x \leq 1$

$\ln(x) < 0$

$\therefore x\ln(x) - x < -1 \quad$ (but since $x < 1$)

$x\ln(x) + 1 < x$

and $\ln(x) + 1/x > 1$

$\therefore \underline{\ln(x) + 1/x \geq 1 \text{ for } x > 0}$

$$g'(x) = \left(x^{x^x}\right)'$$

$$= \left(e^{\ln(x)\cdot x^x}\right)'$$

$$= \left(x^x \cdot \ln(x)\right)' g(x)$$

$$(x^x)' = (\ln(x)+1)x^x$$

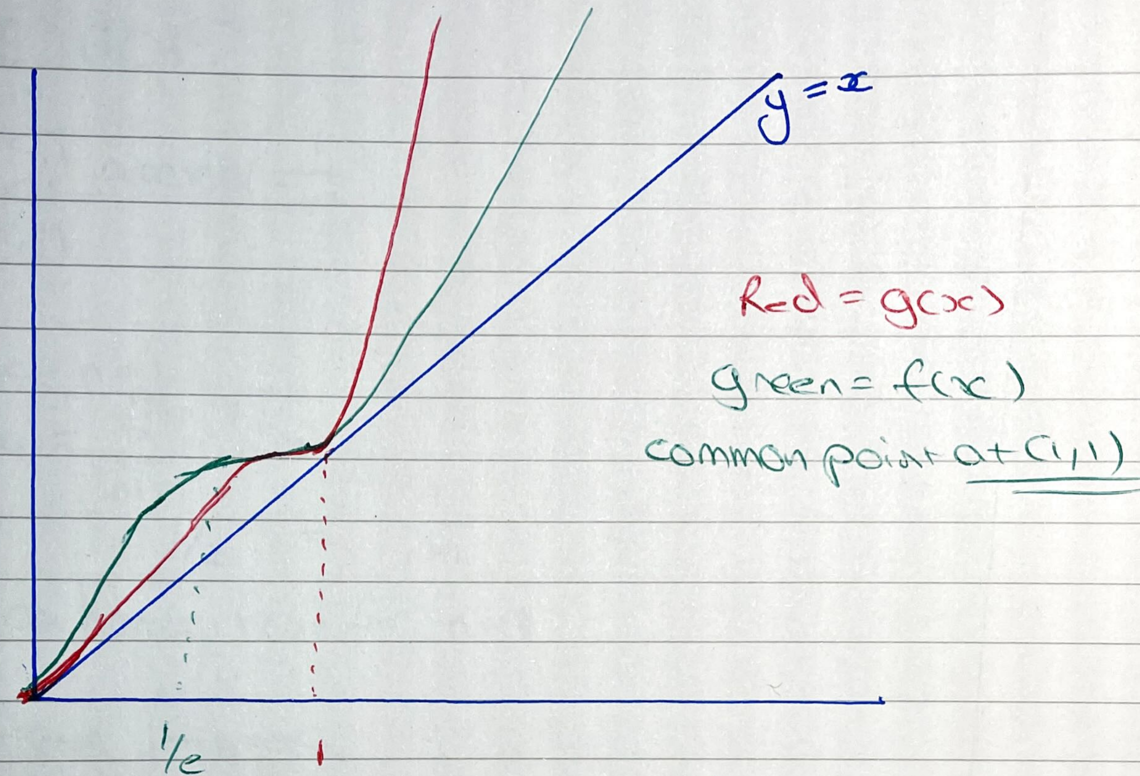
$$(\ln(x))' = 1/x$$

$$\therefore \frac{x^x}{x} + \ln(x)(\ln(x)+1)x^x$$

$$g'(x) \Rightarrow x^x\left(\frac{1}{x} + \ln(x) + \ln^2(x)\right) g(x)$$

$\because 1/x + \ln(x) \geq 1$ and $\ln^2(x) \geq 0$

$\therefore \underline{g'(x) > 0 \text{ for } x > 0}$


y = x
Red = g(x)
green = f(x)
common point at (1,1)
1/e

STEP II 2014

Q6. $\sin(r+{}^1/_2)x - \sin(r-{}^1/_2)x$

$\Rightarrow \sin(rx + x/2) - \sin(rx - {}^1/_2 x)$

$\Rightarrow \sin(rx)\cos(x/2) + \cos(rx)\sin(x/2) - (\sin(rx)\cos(x/2)$
$- \cos(rx)\sin(x/$

$\Rightarrow 2\cos(rx)\sin(\frac{1}{2}x)$

hence $\cos(rx) = \dfrac{\sin(r+{}^1/_2)x - \sin(r-{}^1/_2)x}{2\sin(\frac{1}{2}x)}$

Thus,

$\cos(x) + \cos(2x) \dots \cos(nx)$

$= \dfrac{1}{2\sin({}^1/_2 x)}\Big(-\sin(1-{}^1/_2)x + \sin(1+{}^1/_2)x - \sin(2-{}^1/_2)x$
$\dots - \sin(n-{}^1/_2)x + \sin(n+{}^1/_2)x\Big)$

$\Rightarrow \dfrac{1}{2\sin({}^1/_2 x)}\left(\sin(n+{}^1/_2)x - \sin(\frac{1}{2}x)\right)$

as terms in between get cancelled out!

(i) $S_2(x) = \sin(x) + \dfrac{\sin(2x)}{2}$

hence $(S_n(x))' = \cos(x) + \cos(2x) = 0$

$\therefore 2\cos^2(x) + \cos(x) - 1 = 0$

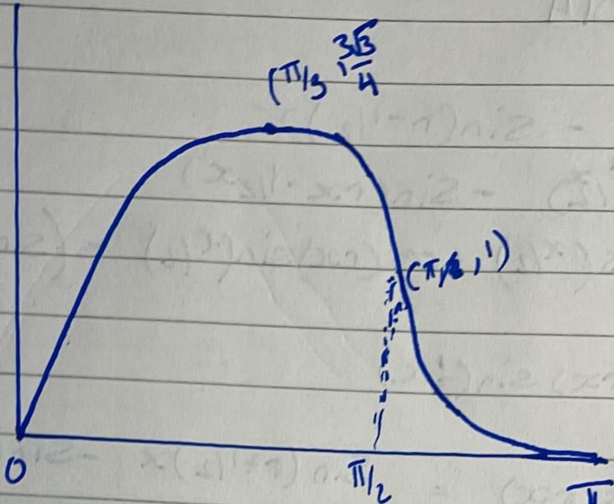
$(2\cos(x) - 1)(\cos(x) + 1) = 0.$

$\cos(x) = -1$

$\therefore x = \pi$

$\cos(x) = 1/2$

$x = \pi/3.$



if S_n has stationary point at $x = x_0$ where $0 < x_0 < \pi$

$$\therefore \quad S_n'(x) = \cos(x) + \cos(2x) \dots \cos(nx)$$

$$= \frac{\sin(n+\tfrac{1}{2})x - \sin(\tfrac{1}{2}x)}{2\sin(\tfrac{1}{2}x)}$$

Which is equal to 0 when $x = x_0$

$$\therefore \quad \frac{\sin(n+\tfrac{1}{2})x_0 - \sin(\tfrac{1}{2}x_0)}{2\sin(\tfrac{1}{2}x_0)} = 0$$

$$\therefore \quad \sin(n+\tfrac{1}{2})x_0 - \sin(\tfrac{1}{2}x_0) = 0$$

$$\therefore \quad \sin(nx_0)\cos(\tfrac{1}{2}x_0) + \cos(nx_0)\sin(\tfrac{1}{2}x_0) - \sin(\tfrac{1}{2}x_0) = 0$$

$$\therefore \quad \sin(nx_0)\cos(\tfrac{1}{2}x_0) = \sin(\tfrac{1}{2}x_0)(1 - \cos(nx_0))$$

$$\therefore \quad \sin(nx_0) = \tan(\tfrac{1}{2}x_0)(1 - \cos(nx_0))$$

$$S_n(x_0) - S_{n-1}(x_0) = \frac{\sin(nx_0)}{n}$$

$$= \frac{\tan(\tfrac{1}{2}x_0)(1 + \cancel{\text{to}}\cos(nx_0))}{n}$$

in $0 < x_0 < \pi$

$\tan(\tfrac{1}{2}x_0) > 0$

and $0 \leq 1-\cos(nx_0) \leq 2$

$\therefore \dfrac{(1-\cos(nx_0))\tan(\tfrac{1}{2}x_0)}{n} \geq 0.$

hence $S_n(x_0) - S_{n-1}(x_0) \geq 0$

$\therefore S_n(x_0) \geq S_{n-1}(x_0)$

$S_n(x) = \dfrac{\sin(nx)}{n} + S_{n-1}(x)$

~~$\sin(nx)$~~) min point of $\dfrac{\sin(nx)}{n}$

$\left(\dfrac{\sin(nx)}{n}\right)' = \cos(nx) = 0$

$\therefore nx = \pi/2, 3\pi/2$

$\left(\dfrac{\sin(nx)}{n}\right)'' = -n\sin(nx)$

$\therefore -\frac{\pi}{2}\sin(\pi/2) = -\pi/2$ it is a max

~~*~~

(iii) ~~S_n~~ if $S_n(x) \geq 0$ (for all x) then $S_{n+1}(x) \geq 0$ for all x in interval

as shown before

Thus for $n=1$, $S_1(x) = \sin(x) \geq 0$ which for $0 \leq x \leq \pi$

hence is true by induction.

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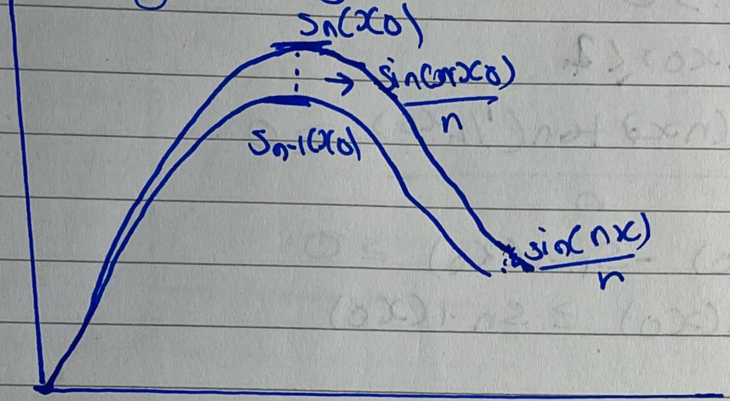
* ~~$f(x_0) > f(x)$~~ $S_n(x) = \sin(x) + \dots \dfrac{\sin(nx)}{n}$

$S_{n-1}(x) = \sin(x) \dots + \dots \dfrac{\sin((n-1)x)}{n-1}$

Illustrating both graphs, we have



and $\left|\frac{\sin(nx)}{n}\right| < \left|\frac{\sin(nx_0)}{n}\right|$

hence $S_n(x) > S_{n-1}(x) > 0$