STEP III 2003

$${}^{2r}C_r = \frac{2r!}{(r!)(r!)} = \frac{(2r!/r!)}{r!}$$

$$= \frac{\frac{1(2)(3)(4)\cdots(2r)}{(1)(2)(3)\cdots(r)}}{r!}$$ → if we multiply of each of these terms by 2, we can divide all the even terms in $2r!$.

r terms

$$\frac{\frac{(1)(2)(3)\cdots(2r)}{(2)(4)\cdots(2r)} \times 2^r}{r!}$$

$$\Rightarrow \frac{1\cdot3\cdot5\cdots 2r-1}{r!} \times 2^r$$ as required.

(i) $$(1-p)^{-1/2} = 1 + \frac{p}{2} + \frac{3}{8}p^2 + \frac{1\cdot3\cdot5}{24}p^3$$

$$= 1 + \frac{1}{2^1}p + \frac{3/2}{2^2}p^2 + \frac{\frac{1\cdot3\cdot5}{6}}{2^3}p^3 \cdots \boxed{\frac{\frac{(1)(3)\cdots(2r-1)}{r!}}{2^r}p^r}$$

↳ this is the general form of any term in this series as the coefficient of p^r is the product of $\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots}{r!}$

$\Rightarrow \frac{\sum \text{sum of odd numbers}}{r! \times 2^r}$ $1/2$ counted r times.

since $${}^{2r}C_r = \frac{(1)(3)(3)\cdots(2r-1)}{r!} \times 2^r$$

$$\therefore \frac{(1)(3)(5)\cdots(2r-1)}{r!} = \frac{{}^{2r}C_r}{2^r}$$

∴ the sequence is now:

$$1 + \frac{{}^2C_1 p}{4^r} + \frac{{}^4C_2}{4^r}p^2 + \frac{{}^8C_3}{4^{3r}}p^3 \cdots \frac{{}^{2n}C_n p^n}{4^{nr}}$$

∴ if we use $p = 1/2$, we obtain

$$1 + \frac{{}^2C_1}{8^r} + \frac{{}^4C_2}{8^{2r}} + \frac{{}^6C_3}{8^{3r}} \cdots$$

$$= \sum_{r=0}^{\infty} \frac{{}^{2r}C_r}{8^r} = (1-1/2)^{-1/2} = \sqrt{2} \quad \square.$$

(ii) Consider $(1-p)^{-3/2}$

$\Rightarrow 1 + 3/2\, p + \frac{3\cdot 5}{8}p^2 + \frac{3\cdot 5\cdot 7}{24}p^3 \ldots$

$\Rightarrow 1 + 3\left(\frac{1}{2}p\right) + 5\left(\frac{3}{8}p^2\right) + 7\left(\frac{3\cdot 5}{24}p^3\right)\ldots$

This is similar to $(1-p)^{-1/2}$ which is $1 + 1/2\, p + 3/8\, p^2 + \frac{3\cdot 5}{24}p^3$

as division by 1 doesn't change the coefficient but for every rth term in this sequence, we have $(2r+1)\underbrace{\left(\frac{{}^{2r}C_r}{4^r}\right)}p^r$

this is established from part 1 and $(2r+1)$ is multiplied to each coefficient as we are starting our product from $-3/2$ instead of $-1/2$.

consider $p = 4/5$

then the sequence $= \sum_{r=0}^{\infty}(2r+1)\left(\frac{{}^{2r}C_r}{4^r}\right)\left(\frac{4}{5}\right)^r$

$= \sum_{r=0}^{\infty}(2r+1)\left(\frac{{}^{2r}C_r}{5^r}\right)$

$= (1-4/5)^{-3/2}$

$= (\sqrt{5})^3 \quad \square.$

$\boxed{29\text{mins}}$.