# STEP II 2017 Q1

Q1. $(n+1)I_n = \int_0^1 (n+1)x^n \arctan(x)\,dx$

Let $\frac{du}{dx} = (n+1)(x^n)$ $\quad v = \arctan(x)$

$u = x^{n+1}$ $\quad \frac{dv}{dx} = 1/x^2+1$

$\therefore (n+1)I_n \int_0^1 = x^{n+1}\arctan(x)\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{x^2+1}\,dx$

$(n+1)I_n = \pi/4 - \int_0^1 \frac{x^{n+1}}{x^2+1}\,dx$

Hence if $n=0$

$\therefore I_0 = \pi/4 - \int_0^1 \frac{x}{x^2+1}\,dx$

$= \pi/4 - \frac{1}{2}\ln(x^2+1)\Big|_0^1$

$= \pi/4 - \frac{1}{2}(\ln(2) - \ln(1))$

$= \pi/4 - \frac{\ln(2)}{2}.$

(ii) $(n+3)I_{n+2} + (n+1)I_n$

$\Rightarrow \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{x^2+1} + \pi/4 - \int_0^1 \frac{x^{n+1}}{x^2+1}\,dx$

$= \frac{\pi}{2} - \int_0^1 x^{n+1}\left(\frac{x^2+1}{x^2+1}\right)dx$

$= \frac{\pi}{2} - \int_0^1 x^{n+1}\,dx$

$= \frac{\pi}{2} - \left[\frac{x^{n+2}}{n+2}\right]_0^1$

$= \pi/2 - 1/n+2$

$$(0+3)I_2 + I_0 = \pi/2 - 1/2$$

$$3I_2 = \pi/2 - 1/2 - \left(\pi/4 - \frac{\ln(2)}{2}\right)$$

$$= \pi/4 + \frac{\ln(2)}{2} - 1/2$$

$$I_2 = \frac{\pi}{12} + \frac{\ln(2)}{6} - 1/6$$

$$\therefore 5I_4 + I_2(3) = \pi/2 - 1/4$$

$$\therefore 5I_4 + \frac{\pi}{4} + \frac{\ln(2)}{2} - \frac{1}{2} = \pi/2 - 1/4$$

$$5I_4 = \pi/4 + 1/4 - \frac{\ln(2)}{2}$$

$$\underline{I_4 = \frac{\pi}{20} + \frac{1}{20} - \frac{\ln(2)}{10}}$$

Base case $n=1$:

has $5I_{4n} = \pi/4 + 1/4 - \frac{\ln(2)}{2}$

$$= \frac{\pi}{4} - \frac{\ln(2)}{2} - \sum_{r=1}^{2} (-1)^r \frac{1}{r}$$

hence $A = \pi/4 - \frac{\ln(2)}{2}$.

TRUE for base case

. Assume true for $k$

. consider case $k+1$:

then $4k+1\, I_{4k} = \pi/4 - \int_0^1 \frac{x^{4k+1}}{x^2+1}\,dx \quad = f(k)$

$$(4k+5)I_{4k+4} = \pi/4 - \int_0^1 \frac{x^{4k+5}}{x^2+1}\,dx = f(k+1)$$

thus $f(k+1) - f(k) = \int_0^1 \frac{x^{4n+1} - x^{4n+5}}{x^2+1} \, dx$

$$= \int_0^1 x^{4n+1} \frac{(1-x^4)}{x^2+1} \, dx$$

$$= \int_0^1 x^{4n+1}(1-x^2) \, dx$$

$$= \left[ \frac{1}{4n+2} - \frac{1}{4n+4} \right]$$

Thus $f(k+1) = A - \frac{1}{2} \sum_{r=1}^{2n} (-1)^r \frac{1}{r} + \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)$

$$= A - \frac{1}{2} \sum_{r=1}^{2n+2} (-1)^r \frac{1}{r}$$

hence true for $k+1$ thus true by Induction.

$\boxed{40\text{mins}}$