STEP II 2017 Q2

$$x_{n+1} = \frac{ax_n - 1}{x_n + b}$$

thus $x_{n+2} = \dfrac{ax_{n+1} - 1}{x_{n+1} + b}$

$$= \frac{a\left(\frac{ax_n - 1}{x_n + b}\right) - 1}{\left(\frac{ax_n - 1}{x_n + b}\right) + b}$$

$$= \frac{a^2x_n - a - (x_n + b)}{ax_n - 1 + bx_n + b^2}$$

$$x_{n+2} = \frac{(a^2-1)x_n - (a+b)}{(a+b)x_n + b^2 - 1}$$

(i) for a sequence to have period 2,

$$x_n = \frac{(a^2-1)x_n - (a+b)}{(a+b)x_n + b^2 - 1}$$

$\therefore (a+b)x_n^2 + b^2 - a^2 x_n + (a+b) = 0$

for any x_n

$\therefore (a+b)(x_n^2 + (b-a)x_n + 1) = 0$

$\therefore \underline{a+b = 0}$

$$x_{n+4} = \frac{a^2-1\left(\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+b^2-1}\right)-(a+b)}{(a+b)\left(\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+b^2-1}\right)+b^2-1}$$

$$= \frac{(a^2-1)^2x_n-(a^2-1)(a+b)-(a+b)^2x_n-(a+b)(b^2-1)}{(a+b)(a^2-1)x_n-(a+b)^2+(b^2-1)(a+b)x_n+(b^2-1)^2}$$

$$= \frac{((a^2-1)^2-(a+b)^2)x_n-(a+b)(a^2+b^2-2)}{(a+b)(a^2+b^2-2)x_n+(b^2-1)^2-(a+b)^2}$$

if $x_n = x_{n+4}$ then period of 4.

thus

$$(a+b)(a^2+b^2-2)x_n^2+((b^2-1)^2-\cancel{(a+b)^2}+\cancel{(a+b)^2}-(a^2-1)^2)x_n + (a+b)(a^2+b^2-2) = 0$$

$$\therefore (a+b)(a^2+b^2-2)x_n^2+(b^2-a^2)(a^2+b^2-2)x_n + a+b(a^2+b^2-2) = 0$$

$$\therefore (a+b)(a^2+b^2-2)(x_n^2+(b-a)x_n+1) = 0$$

$\therefore a+b=0$

or $a^2+b^2=2$

or $x_n = \frac{a-b \pm \sqrt{(b-a)^2-4(1)(1)}}{2}$