STEP 2017 II

Q4: $\left(\int_a^b g(x)\,dx\right)^2 \leq (b-a)\int_a^b g(x)^2\,dx$

Let $b=t,\ a=0 \quad g(x)=e^x$

$\therefore \left(\int_0^t e^x\,dx\right)^2 = (e^t-1)^2 \leq t\underbrace{\int_0^t e^{2x}\,dx}_{\frac{1}{2}(e^{2t}-1)}$

$\therefore \dfrac{(e^t-1)^2}{e^{2t}-1} \leq \dfrac{t}{2}$

$\therefore \dfrac{e^t-1}{e^t+1} \leq \dfrac{t}{2}.$

(ii) Let $b=1,\ a=0$

$\therefore \int_0^1 x g(x)\,dx \leq \frac{1}{3}\int_0^1 (g(x))^2\,dx$

$\therefore (g(x))^2 = e^{-x^2/2}$

$\therefore g(x) = e^{-x^2/4}$

$\therefore 3\int_0^1 x e^{-x^2/4}\,dx \leq \int_0^1 e^{-x^2/2}\,dx$

$3\left(2(1-e^{-1/4})\right)^2 \leq \int_0^1 e^{-x^2/2}\,dx$

$\therefore 12(1-e^{-1/4})^2 \leq \int_0^1 e^{-x^2/2}\,dx$

let $g(x) = \sin^{1/4}(x)$ and $a = 0$

$f(x) = \cos(x)$ $\quad b = \pi/2$

$$\therefore \left(\int_0^{\pi/2} \cos(x)\sin^{1/4}(x)\,dx \right)^2 \leq \left(\int_0^{\pi/2} \cos^2(x)\,dx \right)\left(\int_0^{\pi/2} \sqrt{\sin(x)}\,dx \right)$$

NOTE: $\cos(2x) = 2\cos^2(x) - 1$

$$\therefore \frac{\cos(2x)+1}{2} = \cos^2(x)$$

$$\therefore \frac{1}{2}\int_0^{\pi/2} \cos(2x)+1\,dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{2}\sin(2x) + x\right]_0^{\pi/2}$$

$$\Rightarrow \frac{\pi}{4}$$

$$\therefore \left(\int_0^{\pi/2} \cos(x)\sin^{1/4}(x)\,dx \right)^2 \leq \frac{\pi}{4}\int_0^{\pi/2} \sqrt{\sin(x)}\,dx$$

$$\therefore \left(\frac{4}{5}\left(\sin^{5/4}(x)\right)\Big|_0^{\pi/2} \right)^2 \leq \frac{\pi}{4}\int_0^{\pi/2} \sqrt{\sin(x)}\,dx$$

$$\therefore \frac{64}{25\pi} \leq \int_0^{\pi/2} \sqrt{\sin(x)}\,dx$$

and let $g(x) = \sqrt{\sin(x)}$ $f(x) = 1$ with some bound

$$\therefore \left(\int_0^{\pi/2} \sqrt{\sin(x)}\,dx \right)^2 \leq \left(\frac{\pi}{2}\right)\int_0^{\pi/2} \sin(x)\,dx$$

$$\therefore \left(\int_0^{\pi/2} \sqrt{\sin(x)}\,dx \right)^2 \leq \frac{\pi}{2}\underbrace{\left[-\cos(x)\right]_0^{\pi/2}}_{\boxed{1}}$$

$$\therefore \int_0^{\pi/2} \sqrt{\sin(x)}\,dx \leq \sqrt{\frac{\pi}{2}}$$