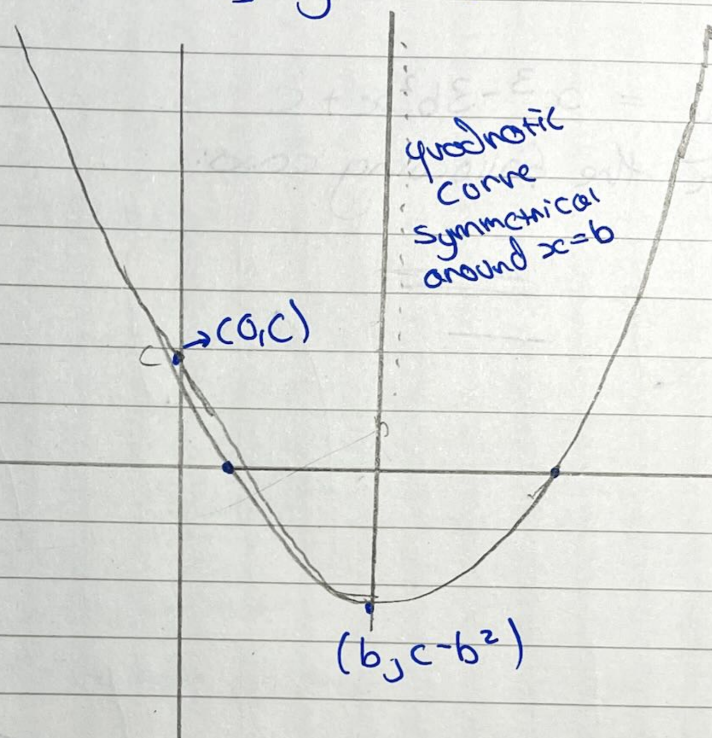
STEP III 2003

Q5. $y = x^2 - 2bx + c \therefore y = (x-b)^2 + c - b^2$

So at min point, $(x-b)^2 = 0$

$\therefore x = b,\ y = c - b^2$



for 2 roots

$\Delta > 0$ so $4b^2 - 4c > 0$

$\therefore b^2 - c > 0$

$\therefore b^2 > c$ (evident from graph as $c - b^2 < 0$ for 2 roots)

if the 2 roots were denoted as R_1, R_2 where $R_1 \leq R_2$
then $\because$ the quadratic graph is symmetrical around $x = b$

$\therefore$ $R_1 = b - m$ where m is some constant

$R_2 = b + m$

now for $R_1, R_2 > 0$ $R_1 > 0$ then $b - m > 0$

$\therefore b > m$

$\therefore \boxed{b+m < 2b}$ hence $\boxed{R_2 < 2b}$

now $R_2 = \dfrac{2b + \sqrt{4b^2 - 4c}}{2}$

$\Rightarrow b + \sqrt{b^2 - c} < 2b$

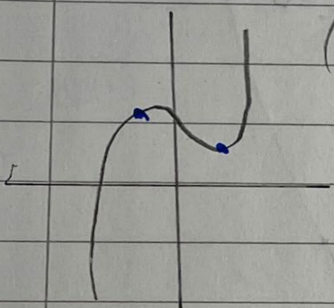
$\therefore \underline{\sqrt{b^2 - c} < b}$

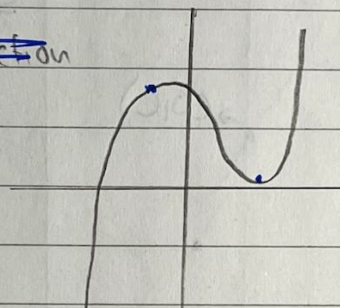
for $y = x^3 - 3b^2 x + c$

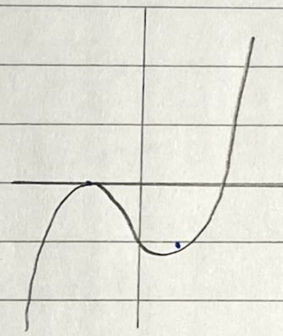
we have the following cases:

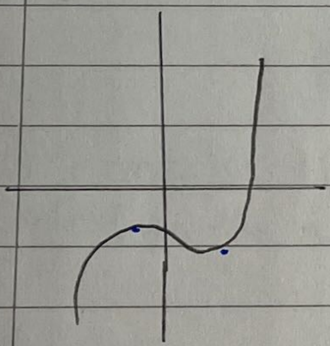
(~~reflection~~

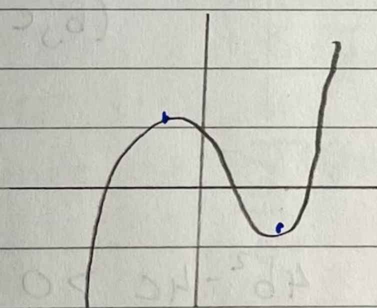
~~on~~











I have labelled the turning points and the only case where ~~where~~ we have 3 distinct solutions is where turning points are strictly above and below 0.

So, $\dfrac{dy}{dx} = 3x^2 - 3b^2 = 0$

$\therefore x^2 = b^2$

$\therefore x = \boxed{\pm b}$

$\therefore y = c - 2b^2$ when $x = b$

$y = c + 2b^2$ when $x = -b$

~~where $b \geq 0$~~

if $c+2b^2 > 0$ and $c-2b^2 < 0$

then $c+2b^2 > 0 > c-2b^2$

$\therefore$ $c+2b^2 > c-2b^2$

$\therefore 2b^2 > -2b^2$

$\therefore 4b^2 > 0$

$b^2 > 0$

$\boxed{\cancel{b>0}}$ $b \neq 0$

and $2b^2 > c > -2b^2$ → necessary ~~if b>0~~

~~if c<0 and b>0, this~~

if $c-2b^2 > 0$ and $c+2b^2 < 0$

$\therefore$ $c-2b^2 > 0 > c+2b^2$

$\therefore 0 > 4b^2$ $\therefore b^2 < 0$

$\therefore \boxed{\cancel{b<0}}$ not possible

and $\underbrace{-2b^2 > c > 2b^2}$

this is not possible as if $b<0$ $\therefore \boxed{b^2 > 0}$

~~for~~ $y = (x-a)^3 - 3b^2(x-a) + c$

must satisfy 2 conditions:

Already shown ← • 3 roots must be distinct and exist

Condition • 3 roots must be positive.

hence equation

Shift the graph by $\binom{a}{0}$

$\therefore$ if R_1, R_2, R_3 were the roots of $y = x^3 - 3b^2x + c$

where $R_1 < R_2 < R_3$

then if $R_1 > -a$

then in the new graph $R_3 > R_2 > R_1 > 0$

→ in other words,

$f(-a) < 0$

$\therefore$ $\underbrace{-a^3 + 3b^2a + c < 0}$

and $\underbrace{2b^2 > c > -2b^2}$

are our 2 conditions

$(x-a)^3 \cancel{\pm} = x^3 - 3ax^2 + 3a^2x \mp a^3$

$\therefore$ new eq $= x^3 - 3ax^2 + 3a^2x - a^3 - 3b^2(x-a) + c$

$= x^3 - 3ax^2 + (3a^2 - 3b^2)x + 3b^2a - a^3 + c$

$2x^3 - 9x^2 + 7x - 1 = 0 = x^3 - \frac{9}{2}x^2 + \frac{7}{2}x - 1/2 = 0$

$\therefore 3a = 9/2$

$\therefore a = 3/2$

$3a^2 - 3b^2 = \frac{7}{2}$

$\therefore 3\left(\frac{9}{4}\right) - 3b^2 = \frac{7}{2} = \frac{27}{4} - 3b^2$

$\therefore 3b^2 = \frac{27}{4} - \frac{7}{2}$

$3b^2 = \frac{13}{4}$

$b^2 = \frac{13}{12}$

$b = \sqrt{\frac{13}{12}}$

and $3b^2a - a^3 + c = -1/2$

$\therefore 3\left(\frac{13}{12}\right)\left(\frac{3}{2}\right) - \left(\frac{27}{8}\right) + c = 1/2$

$\therefore \frac{13}{4}\left(\frac{3}{2}\right) - \frac{27}{8} + c = -1/2$

$\therefore \frac{39-27}{8} + c = -1/2$

$\therefore \frac{12}{8} + c = -1/2 \therefore \boxed{c = 2}$

$\therefore \frac{13}{6} > 2\ 1 > -\frac{13}{6}$ (1st condition passed)

$-\left(\frac{3}{2}\right)^2 + 3b^2a + c = \cancel{-\frac{27}{8}}2 + -1/2 < 0$ (2nd condition passed)

hence the cubic has 3 roots.