Q7.

$$S(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,du \quad t(x) = \int_0^x \frac{1}{1+u^2}\,du$$

$$p = 2\int_0^\infty \frac{1}{1+u^2}\,du$$

(ii) $u = \frac{1}{v}$

$\therefore du = -\frac{1}{v^2}\,dv$ so $t(x) \Rightarrow \int_\infty^{1/x} -\frac{1}{v^2} \cdot \frac{1}{1+(\frac{1}{v})^2}\,dv$

$$\Rightarrow \int_{1/x}^\infty \frac{1}{v^2} \cdot \frac{1}{1+\frac{1}{v^2}}\,dv$$

$$\Rightarrow \int_{1/x}^\infty \frac{1}{v^2} \cdot \frac{v^2}{v^2+1}\,dv$$

$$\Rightarrow \int_{1/x}^\infty \frac{1}{v^2+1}\,dv$$

as v is a dummy variable, this is also equal to $\int_{1/x}^\infty \frac{1}{u^2+1}\,du = t(x)$

therefore $t(x) + t(1/x)$

$$\Rightarrow \int_0^{1/x} \frac{1}{u^2+1}\,du + \int_{1/x}^\infty \frac{1}{u^2+1}\,du$$

$$\Rightarrow \int_0^\infty \frac{1}{u^2+1}\,du = \frac{1}{2}p$$

thus if $x=1$ then

$$t(x) + t(1/x) = t(1) + t(1)$$
$$= 2t(1) = \frac{p}{2}.$$

(ii) $y = \frac{u}{\sqrt{1+u^2}}$ $\quad \therefore y^2 = \frac{u^2}{1+u^2}$

$$\therefore \frac{1}{y^2} = \frac{1+u^2}{u^2} = \frac{1}{u^2} + 1$$

$$\therefore \frac{1}{y^2} - 1 = \frac{1}{u^2}$$

$$\therefore \frac{1-y^2}{y^2} = \frac{1}{u^2} \quad \therefore \frac{y^2}{1-y^2} = u^2$$

hence $u = \pm\sqrt{\frac{y^2}{1-y^2}}$

$$= (\pm)\frac{y}{\sqrt{1-y^2}}$$

~~for $f(x)$, let $y = \frac{u}{\sqrt{1+u^2}}$ and du~~

$$\therefore \frac{du}{dy} = \frac{d}{dy}\left(\frac{y}{(1-y^2)^{1/2}}\right)$$

$$= \frac{1(\sqrt{1-y^2}) - y\left(\frac{1}{2}(1-y^2)^{-1/2}\right)(-2y)}{(1-y^2)}$$

$$= \frac{\sqrt{1-y^2} + \frac{y^2}{\sqrt{1-y^2}}}{(1-y^2)}$$

$$\Rightarrow \frac{1-y^2+y^2}{\sqrt{1-y^2}\cdot(1-y^2)}$$

$$\Rightarrow \left.\frac{1}{(1-y^2)^{3/2}}\right] \quad \square$$

for $f(x)$,

let $y = \frac{u}{\sqrt{1+u^2}}$ and since $\frac{du}{dy} = \frac{1}{(1-y^2)^{3/2}}$

then $du = \cancel{\frac{1}{(1-y^2)^{3/2}}}\,dy$

$$\therefore f(x) \Rightarrow \int_0^{x/\sqrt{1+x^2}} \frac{1}{1+u^2}\cdot\frac{1}{(1-y^2)^{3/2}}\,dy$$

and $\frac{y}{u} = \frac{1}{\sqrt{1+u^2}}$

$$= \int_0^{x/\sqrt{1+x^2}} \left(\frac{y}{u}\right)^2 \left(\frac{1}{(1-y^2)^{3/2}}\right) dy$$

~~and $\frac{dy}{du} = (1-y^2)^{3/2}$~~

$$du = \frac{1}{(1-y^2)^{3/2}}\,dy$$

IGNORE SCRIBALES

RECALL that $u^2 = \dfrac{y^2}{1-y^2}$

$\therefore$ ~~$E(x) = \int_0^{x/\sqrt{x^2+1}} \frac{y^2/(1-y^2)}{y^2} \cdot \frac{1}{(1-y)^{3/2}} \, dy$~~

$$E(x) \Rightarrow \int_0^{x/\sqrt{x^2+1}} \frac{y^2}{\left(y^2/1-y^2\right)} \cdot \frac{1}{(1-y^2)^{3/2}} \, dy$$

$$\Rightarrow \int_0^{x/\sqrt{x^2+1}} \frac{1}{\sqrt{1-y^2}} \, dy = S\left(\frac{x}{\sqrt{x^2+1}}\right)$$

$$\therefore S\left(\frac{1}{\sqrt{1+1}}\right) = S\left(\frac{1}{\sqrt{2}}\right) = E(1) = \frac{1}{2}\left(\frac{1}{2}P\right)$$
$$= \frac{1}{4}P.$$

(iii) $$Z = \frac{U + 1/\sqrt{3}}{1 - 1/\sqrt{3}\, U}$$

$$\therefore Z - \frac{1}{\sqrt{3}}U(Z) = U + 1/\sqrt{3}$$

$$\therefore \frac{Z - 1/\sqrt{3}}{\cancel{Z + 1/\sqrt{3}}} = U\left(1 + 1/\sqrt{3}\, Z\right)$$

$$\therefore \frac{Z - 1/\sqrt{3}}{1 + 1/\sqrt{3}\, Z} = U$$

$$\therefore \frac{\left(1 + 1/\sqrt{3}\, Z\right) - 1/\sqrt{3}\left(Z - 1/\sqrt{3}\right)}{\left(1 + 1/\sqrt{3}\, Z\right)^2} = \frac{dU}{dZ}$$

hence ~~$E(1/\sqrt{3})$~~ $E(x) =$ $$\frac{1 + 1/\sqrt{3}\,Z - 1/\sqrt{3}\, Z + 1/3}{\left(1 + 1/\sqrt{3}\, Z\right)^2} = \frac{4/3}{\left(1 + 1/\sqrt{3}\, Z\right)^2}$$

$$\therefore E(x) = \int_{1/\sqrt{3}}^{\frac{x + 1/\sqrt{3}}{1 - \frac{1}{\sqrt{3}}x}} \frac{1}{\frac{(Z - 1/\sqrt{3})^2}{(1 + \frac{1}{\sqrt{3}}Z)^2} + 1} \cdot \frac{\frac{4}{3}}{\left(1 + \left(1/\sqrt{3}\, Z\right)\right)^2} \, dZ$$

let $U_P = \dfrac{x + 1/\sqrt{3}}{1 - 1/\sqrt{3}\,x}$

$$t(x) = \int_{1/\sqrt{3}}^{\frac{x + 1/\sqrt{3}}{1 - 1/\sqrt{3}x}} \frac{4}{3} \cdot \frac{1}{(z - 1/\sqrt{3})^2 + (1/\sqrt{3}\,z + 1)^2} \, dz$$

$$= 4/3 \int_{1/\sqrt{3}}^{U_P} \frac{1}{z^2 - 2/\sqrt{3}\,z + 1/3 + \frac{1}{3}z^2 + 2/\sqrt{3} + 1} \, dz$$

$$= \frac{4}{3} \int_{1/\sqrt{3}}^{U_P} \frac{1}{\frac{4}{3}z^2 + \frac{4}{3}} \, dz$$

$$= \int_{1/\sqrt{3}}^{U_P} \frac{1}{z^2 + 1} \, dz$$

$$\therefore t(1/\sqrt{3}) = \int_{1/\sqrt{3}}^{\frac{1/\sqrt{3} + 1/\sqrt{3}}{1 - 1/3}} \frac{1}{z^2 + 1} \, dz$$

$$= \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{z^2 + 1} \, dz$$

$\therefore$ $t(\sqrt{3}) + t(1/\sqrt{3}) = 1/2 P$ (from part i)

<u>NTS:</u> $t(\sqrt{3}) = 2(t(1/\sqrt{3}))$

but $t(\sqrt{3}) = \displaystyle\int_0^{\sqrt{3}} \frac{1}{1+z^2} \, dz = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+z^2} \, dz + \int_0^{1/\sqrt{3}} \frac{1}{1+z^2} \, dz$

$= t(1/\sqrt{3}) + t(1/\sqrt{3})$

$\square$.

hence $t(\sqrt{3}) = 2t(1/\sqrt{3})$

$\therefore$ $t(\sqrt{3}) + t(1/\sqrt{3}) = 3t(1/\sqrt{3}) = 1/2 P$