STEP III 2000. Q2

$$\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{1/2} dx$$

when $x = 2-\cos\theta$
then $\cos\theta = 2-x$
$\therefore -\sin\theta\, d\theta = -1\, dx$
$\therefore \sin\theta\, d\theta = dx$

$$\therefore \int_{\pi/3}^{\pi/2} \left(\frac{(2-\cos\theta)-1}{3-(2-\cos\theta)}\right)^{1/2} \cdot \sin\theta\, d\theta$$

$$\Rightarrow \int_{\pi/3}^{\pi/2} \left(\frac{1-\cos\theta}{\cos\theta+1}\right)^{1/2} \sin\theta\, d\theta$$

$$\Rightarrow \int_{\pi/3}^{\pi/2} \left(\frac{1-\cos^2\theta}{(\cos\theta+1)^2}\right)^{1/2} \sin\theta\, d\theta \quad (\text{and as } 1-\cos^2\theta = \sin^2\theta)$$

$$\Rightarrow \int_{\pi/3}^{\pi/2} \left(\frac{\sin^2(\theta)}{(\cos\theta+1)^2}\right)^{1/2} \sin\theta\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta+1}\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{1-\cos^2\theta}{\cos\theta+1}\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{(1+\cos\theta)(1-\cos\theta)}{\cos\theta+1}\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} (1-\cos\theta)\, d\theta$$

$$\Rightarrow \left[\theta - \sin\theta\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi}{2} - 1\right) - \left(\pi/3 - \sqrt{3}/2\right)$$
$$= \pi/6 - 1 + \sqrt{3}/2.$$

28 mins

Let $x = \frac{b+a}{2} - \frac{b-a}{2}\cos\theta$

then $dx = \frac{b-a}{2}\sin\theta\, d\theta$

Upper bound $= \frac{a+b}{2} \quad \therefore \theta = \frac{\pi}{2}$

$P = \frac{3a+b}{4} \quad \therefore \frac{b-a}{2}\cos\theta = \frac{b+a}{2} - \frac{3a+b}{4}$

$= \frac{2b+2a-(3a+b)}{4}$

$= \frac{b-a}{4}$

$\therefore \cos\theta = 1/2$

$\therefore \theta = \pi/3$

$$\therefore \frac{b-a}{2}\int_{\pi/3}^{\pi/2}\left(\frac{\frac{b+a}{2} - \frac{b-a}{2}\cos\theta - a}{b-\left(\frac{b+a}{2} - \frac{b-a}{2}\cos\theta\right)}\right)^{1/2}\sin\theta\, d\theta$$

$$\Rightarrow \frac{b-a}{2}\int_{\pi/3}^{\pi/2}\left(\frac{\frac{b-a}{2} - \frac{b-a}{2}\cos\theta}{\frac{b-a}{2} + \frac{b-a}{2}\cos\theta}\right)^{1/2}\sin\theta\, d\theta$$

$$\Rightarrow \frac{b-a}{2}\int_{\pi/3}^{\pi/2}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}\sin\theta\, d\theta$$

$$\Rightarrow \frac{b-a}{2}\int_{\pi/3}^{\pi/2} 1-\cos\theta\, d\theta \quad (\text{using point 1})$$

$$\Rightarrow \frac{b-a}{2}\left(\frac{\pi}{6} - 1 - \frac{\sqrt{3}}{2}\right) = \frac{b-a\left(\pi - 6 + 3\sqrt{3}\right)}{12}$$

as required