STEP III 2012 Q3.

(i) $(1-x)(1+x)(1+x^2)\ldots(1+x^{2^n})$

$\Rightarrow \underbrace{(1-x^2)(1+x^2)}_{\text{difference of 2 squares}}(1+x^4)\ldots(1+x^{2^n})$

$\hookrightarrow (1-x^4)(1+x^4)\ldots(1+x^{2^n})$

and this method repeats ~~under we~~ until

$(1-x^{2^n})(1+x^{2^n}) = \underline{1-x^{2^{n+1}}}$

consider $\dfrac{1-x^{2^{n+1}}}{1-x}$

$$= \frac{(1-x)(1+x)(1+x^2)\ldots(1+x^{2^n})}{(1-x)} \quad \text{(as shown in part (a))}$$

$$= (1+x)(1+x^2)\ldots(1+x^{2^n})$$

$$\therefore \frac{1}{1-x} - \frac{x^{2^{n+1}}}{1-x} = (1+x)(1+x^2)\ldots(1+x^{2^n})$$

$$\therefore \frac{1}{1-x} = (1+x)(1+x^2)\ldots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x} \quad \square$$

Unravelling backwards,

$$\frac{1-x^{2^{n+1}}}{1-x} = (1+x)(1+x^2)\ldots(1+x^{2^n})$$

$$\therefore \ln\left(\frac{1-x^{2^{n+1}}}{1-x}\right) = \ln((1+x)(1+x^2)\ldots + (1+x^{2^n}))$$

$$\therefore \ln(1-x^{2^{n+1}}) - \ln(1-x) = \ln(1+x) + \ln(1+x^2) + \ldots \ln(1+x^{2^n})$$

$$-\ln(1-x) = \ln(x+1) + \ln(x^2+1) + \dots \ln(x^{2^n}+1) \quad - \ln(1-x^{2^{n+1}})$$

$$\therefore \ln(1-x) = -\left(\ln(x+1) + \ln(x^2+1) \dots + \ln(x^{2^n}+1)\right) + \ln(1-x^{2^{n+1}})$$

Since $\frac{1}{1-x}$ contains an infinite number of terms from the general binomial theorem, it follows that n should tend to infinity. ~~to find the root value~~

$$\therefore \ln(1-x) = -\sum_{r=1}^{\infty} \ln(x^{2^r}+1) + \lim_{n\to\infty}\left(\ln(1-x^{2^{n+1}})\right)$$

$$= -\underbrace{\sum_{r=0}^{\infty} \ln(x^{2^r}+1)}_{} + \ln\left(\lim_{n\to\infty}(1-x^{2^{n+1}})\right)$$

as $\ln(x^0+1)$ $= \ln(1)$ $= 0$, therefore value of sum is not changed.

$\because |x| < 1$

$\therefore \lim_{n\to\infty}(x^{2^{n+1}}) \mapsto 0$

$\therefore \lim_{n\to\infty}(1-x^{2^{n+1}}) \mapsto 1$

and $\ln(1) = 0$

$$\therefore \ln(1-x) = -\sum_{r=1}^{\infty} \ln(x^{2^r}+1)$$

$$\frac{d}{dx}(\ln(1-x)) = -\left(\frac{d}{dx}\left(\sum_{r=1}^{\infty} \ln(x^{2^r}+1)\right)\right)$$

$$= \frac{-1}{1-x} = -1\left(\frac{1}{x+1} + \frac{2x}{1+x^2} \dots\right)$$

$$\therefore \frac{1}{1-x} = \frac{1}{x+1} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^2} \dots$$

consider:

$$\underbrace{\underbrace{(1+x+x^2)(1-x+x^2)}_{(1+x^4+x^2)}(1-x^2+x^4)}_{(1+x^8+x^4)} \cdots (1-x^{2^n}+x^{2^{n+1}})$$

↳ and this method repeats via (difference of 2 squares) to yield $(1+x^{2^n}+x^{2^{n+1}})(1-x^{2^n}+x^{2^{n+1}})$

$$= \left((1+x^{2^{n+1}})^2 - x^{2^{n+1}}\right)$$

$$= 1+x^{2^{n+1}}+x^{2^{n+2}}$$

and therefore

$$\frac{1+x^{2^{n+1}}+x^{2^{n+2}}}{1+x+x^2} = (1-x+x^2)(1-x^2+x^4)\cdots(1-x^{2^n}+x^{2^{n+1}}) \quad (+)$$

∴ ln both sides yields.

$$\ln\left(1+x^{2^{n+1}}+x^{2^{n+2}}\right) - \ln(1+x+x^2) = \sum_{r=1}^{\cancel{\infty}\, n} \ln(1-x^{2^r}+x^{2^{r+1}})$$

$$= \sum_{r=0}^{\cancel{\infty}\, n} \ln\left(1-x^{2^r}+x^{2^{r+1}}\right)$$

as $\ln(1-\cancel{x}+1) = \ln(1) = 0$

(∴ won't impact RHS)

∴ Since $\frac{1}{1+x+x^2}$ has infinite number of terms, from general binomial expansion; n has to tend to infinity

$$\lim_{n\to\infty}\left(\ln(\underbrace{1+x^{2^{n+1}}+x^{2^{n+2}}}_{|x|<1})\right) - \sum_{r=0}^{\infty}\left(\ln(1-x^{2^r}+x^{2^{r+1}})\right) = \ln(1+x+x^2)$$

∴ this tends to 0

∴ the expression tends to $\ln(1-0) = \underline{\ln(1) = 0}$

$$\therefore -\sum_{r=0}^{\infty} \ln\left(1 - x^{2^r} + x^{2^{r+1}}\right) = \ln(1+x+x^2)$$

ifferentiate both sides and,

$$-\left(\frac{2x-1}{x^2-x+1} + \frac{4x^3-2x}{x^4-x^2+1} \cdots\cdots\right) = \frac{1+2x}{1+x+x^2}$$

$$\therefore \frac{1+2x}{1+x+x^2} = \frac{1-2x}{x^2-x+1} + \frac{2x-4x^3}{x^4-x^2+1} \cdots\cdots \quad \square$$

30mins